RESEARCH NOTES

A Note on the Construction of the $(2^{16}, 2^{11})$ and Other Associated Confounded Designs, Keeping up to Second Order Interactions Unconfounded

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1. Introduction

Using the geometrical theory of confounding, Bose (1947) has investigated the problem of determining the maximum number of factors that can be accommodated in the case of the general symmetrical factorial design s^m, in which each block is of size s^r, so that no degrees of freedom belonging to a main effect or an interaction involving t or a lesser number of factors are confounded. Denoting this number by $m_t(r, s)$, he has shown that this equals the maximum number of points which can be chosen in $PG(r-1, s)$ so that no 't' should be conjoint. By noting that $m_2(r, s)$ must equal the number of distinct points in $PG(r-1, s)$, we obtain Fisher's result (Fisher, 1942, 1945)

$$m_2(r, s) = \frac{s^r - 1}{s - 1}. \tag{1}$$

For the case $t = 3$, Bose has obtained the following results:—

(*a*) $m_3(3, s) = s + 2$, when s is a power of 2.

$= s + 1$, when s is a power of an odd prime.

(*b*) $m_3(4, s) = s^2 + 1$, when s is a power of an odd prime.

(*c*) $m_3(r, 2) = 2^{r-1}$.

It follows from (*c*) that in a factorial experiment in which each factor is at 2 levels and the block size is 2^5, the maximum number of factors that can be accommodated so that no main effect or first order interaction is confounded is 2^4. In this note, we shall, using Bose's method, construct the $(2^{16}, 2^{11})$ and the associated $(2^{15}, 2^{10})$, $(2^{14}, 2^9)$, etc., designs, and also enumerate the degrees of freedom actually confounded in each case.

2. The $(2^{16}, 2^{11})$ Confounded Design

According to Bose's method, the construction of the $(2^{16}, 2^{11})$ design, keeping up to second order interactions unconfounded, requires the selection of 16 points in $PG(4, 2)$, no three of which are collinear. These are given by the columns of the matrix.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \end{bmatrix} \qquad (2)$$

The confounded design corresponds to a 4-flat in the 15-flat at infinity in $PG(16, 2)$, and the equations of this 4-flat at infinity can be written down in the form

$$\left.\begin{aligned} x_1 + x_2 + x_3 + x_6 &= 0 \\ x_1 + x_2 + x_4 + x_7 &= 0 \\ x_1 + x_2 + x_5 + x_8 &= 0 \\ x_1 + x_3 + x_4 + x_9 &= 0 \\ x_1 + x_4 + x_5 + x_{10} &= 0 \\ x_1 + x_3 + x_5 + x_{11} &= 0 \\ x_2 + x_3 + x_4 + x_{12} &= 0 \\ x_3 + x_4 + x_5 + x_{13} &= 0 \\ x_2 + x_4 + x_5 + x_{14} &= 0 \\ x_2 + x_3 + x_5 + x_{15} &= 0 \\ x_1 + x_2 + x_3 + x_4 + x_5 + x_{16} &= 0 \end{aligned}\right\} \quad x_0 = 0 \qquad (3)$$

In terms of the theory of groups, (3) would correspond to the 11 generators *ABCF, ABDG, ABEH, ACDJ, ADEK, ACEL, BCDM, CDEN, BDEO, BCEP* and *ABCDEQ* of the confounding sub-group.

If $c_1, c_2, \ldots, c_{11}$ be any elements of $GF(2)$, the treatments in the corresponding block are determined by the equations:

$$\left.\begin{aligned}
x_1 + x_2 + x_3 + x_6 &= c_1 \\
x_1 + x_2 + x_4 + x_7 &= c_2 \\
x_1 + x_2 + x_5 + x_8 &= c_3 \\
x_1 + x_3 + x_4 + x_9 &= c_4 \\
x_1 + x_4 + x_5 + x_{10} &= c_5 \\
x_1 + x_3 + x_5 + x_{11} &= c_6 \\
x_2 + x_3 + x_4 + x_{12} &= c_7 \\
x_3 + x_4 + x_5 + x_{13} &= c_8 \\
x_2 + x_4 + x_5 + x_{14} &= c_9 \\
x_2 + x_3 + x_5 + x_{15} &= c_{10} \\
x_1 + x_2 + x_3 + x_4 + x_5 + x_{16} &= c_{11}
\end{aligned}\right\} \tag{4}$$

The intrablock sub-group is given by the 32 treatment combination in the block obtained by taking $c_1 = c_2 = \ldots = c_{11} = 0$, and is shown in Table I.

Taking the 2048 possible values of $c_1, c_2, \ldots, c_{11}$, we get the 2048 blocks of the design. The 2047 degrees of freedom confounded belong to the 2047 pencils represented by

$$\begin{aligned} P\{&\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6 + \lambda_{11},\ \lambda_1 + \lambda_2 + \lambda_3 + \lambda_7 + \lambda_9 \\ &+ \lambda_{10} + \lambda_{11},\ \lambda_1 + \lambda_4 + \lambda_6 + \lambda_7 + \lambda_8 + \lambda_{10} + \lambda_{11}, \lambda_2 \\ &+ \lambda_4 + \lambda_5 + \lambda_7 + \lambda_8 + \lambda_9 + \lambda_{11},\ \lambda_3 + \lambda_5 + \lambda_6 + \lambda_8 + \\ &\lambda_9 + \lambda_{10} + \lambda_{11},\ \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8, \lambda_9, \lambda_{10}, \lambda_{11}\} \end{aligned} \tag{5}$$

where the λ's do not simultaneously vanish. Each pencil now carries only one degree of freedom, and calculations made show that of the 2047 degrees of freedom confounded, 140 belong to 4-factor interactions, 448 to 6-factor interactions, 870 to 8-factor interactions, 448 to 10-factor interactions, 140 to 12-factor interactions and the remaining one to the 16-factor interaction.

3. The $(2^{15}, 2^{10})$ Confounded and other Associated Designs

For obtaining a $(2^{15}, 2^{10})$ design of the required type, that is, not confounding any main effect or first or second order interaction, we have to simply start with the matrix obtained from (2) by omitting one column, say, the last. Thus, the columns of the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \end{bmatrix} \quad (6)$$

give the co-ordinates of 15 points in $PG(4, 2)$, no three of which are collinear.

Treatments are now represented by the finite points of $PG(15, 2)$, and the vertex of the required design is given by the equations (3), omitting the last equation, *viz.*, $x_1 + x_2 + x_3 + x_4 + x_5 + x_{16} = 0$, so that the 10 generators of the confounding sub-group are *ABCF*, *ABDG*, *ABEH*, *ACDJ*, *ADEK*, *ACEL*, *BCDM*, *CDEN*, *BDEO* and *BCEP*.

The degrees of freedom confounded are carried by the 1023 pencils represented by

$$\begin{aligned} P\{&\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6, \lambda_1 + \lambda_2 + \lambda_3 + \lambda_7 + \lambda_9 + \lambda_{10}, \\ &\lambda_1 + \lambda_4 + \lambda_6 + \lambda_7 + \lambda_8 + \lambda_{10}, \lambda_2 + \lambda_4 + \lambda_5 + \\ &\lambda_7 + \lambda_8 + \lambda_9, \lambda_3 + \lambda_5 + \lambda_6 + \lambda_8 + \lambda_9 + \lambda_{10}, \lambda_1, \lambda_2, \\ &\lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8, \lambda_9, \lambda_{10}\}. \end{aligned} \quad (7)$$

In this case, of the 1023 degrees of freedom confounded, 105 belong to 4-factor interactions, 280 to 6-factor interactions, 435 to 8-factor interactions, 168 to 10-factor interactions and 35 to 12-factor interactions.

Following this procedure, the $(2^{14}, 2^9)$, $(2^{13}, 2^8)$, $(2^{12}, 2^7)$, $(2^{11}, 2^6)$, $(2^{10}, 2^5)$, $(2^9, 2^4)$, $(2^8, 2^3)$, $(2^7, 2^2)$ and $(2^6, 2)$ designs are readily constructed, the number of degrees of freedom confounded for the various high order interactions in these designs being as shown in Table II, in which the degrees of freedom confounded for the $(2^{16}, 2^{10})$ and $(2^{15}, 2^{10})$ have also been shown for convenience of reference.

TABLE I

Intrablock sub-group for $((2^{16}, 2^{11})$ *confounded design*

Co-ordinates of 32 points lying on the finite 5-flat																Corresponding treatment combinations
x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}	x_{12}	x_{13}	x_{14}	x_{15}	x_{16}	
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
1	0	0	0	0	1	1	1	1	1	1	0	0	0	0	1	*a f g h j k l q*
0	1	0	0	0	1	1	1	0	0	0	1	0	1	1	1	*b f g h m o p q*
1	1	0	0	0	0	0	0	1	1	1	1	0	1	1	0	*a b j k l m o p*
0	0	1	0	0	1	0	0	1	0	1	1	1	0	1	1	*c f j l m n p q*
1	0	1	0	0	0	1	1	0	1	0	1	1	0	1	0	*a c g h k m n p*
0	1	1	0	0	0	1	1	1	0	1	0	1	1	0	0	*b c g h j l n o*
1	1	1	0	0	1	0	0	0	1	0	0	1	1	0	1	*a b c f k n o q*
0	0	0	1	0	0	1	0	1	1	0	1	1	1	0	1	*d g j k m n o q*
1	0	0	1	0	1	0	1	0	0	1	1	1	1	0	0	*a d f k l m n o*
0	1	0	1	0	1	0	1	1	1	0	0	1	0	1	0	*b d f g h j n p*
1	1	0	1	0	0	1	0	0	0	1	0	1	0	1	1	*a b d g l n p q*
0	0	1	1	0	1	1	0	1	1	1	0	0	1	1	0	*c d f g j k l o p*

1	0	1	1	0	0	0	1	0	0	0	0	0	1	1	1	*a c d h o p* [illegible]
0	1	1	1	0	0	0	1	1	1	1	1	0	0	0	1	*b c d h j k* [illegible]
1	1	1	1	0	1	1	0	0	0	0	1	0	0	0	0	*a b c d f g m*
0	0	0	0	1	0	0	1	0	1	1	0	1	1	1	1	*e h k l n o p q*
1	0	0	0	1	1	1	0	1	0	0	0	1	1	1	0	*a e f g j n o p*
0	1	0	0	1	1	1	0	0	1	1	1	1	0	0	0	*b e f g k l m n*
1	1	0	0	1	0	0	1	1	0	0	1	1	0	0	1	*a b e h j m n q*
0	0	1	0	1	1	0	1	1	1	0	1	0	1	0	0	*c e f h j k m o*
1	0	1	0	1	0	1	0	0	0	1	1	0	1	0	1	*a c e g l m o q*
0	1	1	0	1	0	1	0	1	1	0	0	0	0	1	1	*b c e g j k p q*
1	1	1	0	1	1	0	1	0	0	1	0	0	0	1	0	*a b c e f h l p*
0	0	0	1	1	0	1	1	1	0	1	1	0	0	1	0	*d e g h i l m p*
1	0	0	1	1	1	0	0	0	1	0	1	0	0	1	1	*a d e f k m p q*
0	1	0	1	1	1	0	0	1	0	1	0	0	1	0	1	*b d e f j l o q*
1	1	0	1	1	0	1	1	0	1	0	0	0	1	0	0	*a b d e g h k o*
0	0	1	1	1	1	1	1	0	0	0	0	1	0	0	1	*c d e f g h n q*
1	0	1	1	1	0	0	0	1	1	1	0	1	0	0	0	*a c d e j k l n*
0	1	1	1	1	0	0	0	0	0	0	1	1	1	1	0	*b c d e m n o p*
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	*a b c d e f g h j k l m n o p q*

TABLE II

Number of degrees of freedom for various high order interactions confounded in (2^{16}, 2^{11}), *etc., confounded designs*

Design	Total No. of degrees of freedom confounded	Degrees of freedom confounded for interactions of						
		4 Factors	6 Factors	8 Factors	10 Factors	12 Factors	14 Factors	16 Factors
(2^{16}, 2^{11})	2047	140	448	870	448	140	0	1
(2^{15}, 2^{10})	1023	105	280	435	168	35	0	0
(2^{14}, 2^{9})	511	77	168	203	56	7	0	0
(2^{13}, 2^{8})	255	55	96	87	16	1	0	0
(2^{12}, 2^{7})	127	38	52	33	4	0	0	0
(2^{11}, 2^{6})	63	25	27	10	1	0	0	0
(2^{10}, 2^{5})	31	16	12	3	0	0	0	0
(2^{9}, 2^{4})	15	10	4	1	0	0	0	0
(2^{8}, 2^{3})	7	6	0	1	0	0	0	0
(2^{7}, 2^{2})	3	3	0	0	0	0	0	0
(2^{6}, 2)	1	1	0	0	0	0	0	0

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1. Bose, R. C. .. "Mathematical theory of the symmetrical factorial design," *Sankhya*, 1947, **8**, 107–66.
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3. ——— .. "A system of confounding for factors with more than two alternatives, giving completely orthogonal cubes and higher powers," *ibid.*, 1945, **12**, 283–90.

Note on Two Papers of K. R. Nair

BY H. D. BRUNK

CERTAIN questions raised by Dr. K. R. Nair in recent papers[1,2] and in correspondence with the author are answered in this note. Corollaries 1 and 2 below are elementary in character, and may appear in one form or another in the literature. Nevertheless it is hoped that their presentation here will be of interest to readers of Dr. Nair's papers.

Let $x_1, \ldots, x_n$ be real numbers, $x_1 \leqslant x_2 \leqslant \ldots, \leqslant x_n$, $\bar{x} = \sum_{i=1}^{n} x_i/n$, $S_x^2 = \sum_{i=1}^{n} (x_i - \bar{x})^2/(n-1)$. The questions are as to the smallest possible values of certain ratios of the form $L(x_1, \ldots, x_n)/S_x$, where in each case $L(x_1, \ldots, x_n)$ is a linear combination of $x_1, x_2, \ldots, x_n$:

$$L = \bar{x} - x_1; \quad (1)$$

$$L = \Sigma \Sigma (x_i - x_j) \; (1 \leqslant i < j \leqslant n); \quad (2)$$

$$L = x_n - x_1. \quad (3)$$

We may suppose without loss of generality that $x_1 = 0$, $x_n = 1$, since the ratio L/S_x is invariant under translation and change of scale.

Lemma.—If Z *is a random variable distributed over* [0, 1], *if* $2c > a > 0$, $h > 0$, *then*

$$[a + hEZ]^2/[c + hEZ^2] \geqslant a^2/c,$$

with equality when $Z = 0$ *almost surely.*

Proof.—This is an immediate consequence of $EZ^2 \leqslant EZ$ (EZ denotes the expectation of Z).

Corollary 1.—*If* X *is a random variable distributed over* [0, 1], *if* $\Pr\{X = 1\} \geqslant p$ *then*

$$pEX^2 \leqslant [EX]^2,$$

with equality when $\Pr\{X = 0\} = q = 1 - p$, $\Pr\{X = 1\} = p$.

Proof.—The distribution of X may be achieved in the following way. Let T be a random variable distributed over [0, 1], and let $X = T$ with probability $1 - p = q$, $X = 1$ with probability p. Then $EX = p + qET$, $EX^2 = p + qET^2$. It suffices to apply the lemma with $a = c = p$, $h = q$.

Corollary 1 a.—$(\bar{x} - x_1)^2/S_x^2 \geqslant 1/n$, *with equality when* $x_1 = x_2 = \ldots = x_{n-1}$.

Proof.—Apply Corollary 1 to the random variable X which takes each of the values $x_1, x_2, \ldots, x_n$ with probability $1/n$; $p = 1/n$. Then $EX = \bar{x}$, $VX = (n-1) S_x^2/n$. By Corollary 1, $[EX^2]/[EX]^2 \leqslant 1/p$, hence $[VX]/[EX]^2 = \{EX^2 - [EX]^2\}/[EX]^2 \leqslant 1/p - 1 = q/p$, so that $[EX^2]/[VX] \geqslant p/q$ and $\bar{x}^2/[(n-1) S_x^2/n] \geqslant (1/n)/(1 - 1/n)$

$= 1/(n-1)$. Thus $\bar{x}^2/S_x^2 \geqslant 1/n$, with equality when $x_2 = x_3 = \ldots = x_{n-1} = 0$ (assuming $x_1 = 0$, $x_n = 1$).

One finds by applying Corollary 1 to $1 - X$ that also $(x_n - \bar{x})^2/S_x^2 \geqslant 1/n$, *with equality when* $x_2 = \ldots = x_n$.

Corollary 2.—*If* X, Y *are independent random variables identically distributed over* [0, 1] *and if* $\Pr\{X = 0\} \geqslant p$, $\Pr\{X = 1\} \geqslant p$ $(p < \frac{1}{2})$ *then* $[E\,|\,Y - X\,|]^2/[VX] \geqslant 4pq$, *with equality when* $\Pr\{X = 0\} = q = 1 - p$, $\Pr\{X = 1\} = p$, *or when* $\Pr\{X = 0\} = p$, $\Pr\{X = 1\} = q$.

Proof.—The joint distribution of X and Y is achieved by introducing independent random variables T and U, identically distributed over [0, 1], and letting $X = T$ with probability $1 - 2p = q - p$, $X = 0$ with probability p, $X = 1$ with probability p; $Y = U$ with probability $q - p$, $Y = 0$ with probability p, $Y = 1$ with probability p. Straightforward calculation yields

$$E\,|\,Y - X\,| = 2pq + (q-p)^2 E\,|\,U - T\,|,$$

$$E\,|\,Y - X\,|^2 = 2p\,\{p + (q-p)\,[E\,(1-T)^2 + ET^2]\} + (q-p)^2 E\,(U-T)^2.$$

Apply the lemma, with $a = 2pq$,

$$c = 2p\,\{p + (q-p)\,[E\,(1-T)^2 + ET^2]\},\ h = (q-p)^2;$$

observing that $2c > a$. One has

$$[E\,|\,Y - X\,|]^2/[E\,|\,Y - X\,|^2] \geqslant a^2/c \geqslant 2pq,$$

since $E\,(1-T)^2 + E\,(T^2) \leqslant 1$; equality is achieved when $T \equiv 0$ or $T \equiv 1$, $U \equiv T$. Since $E\,(Y-X)^2 = 2VX$, one has the conclusion of Corollary 2.

Corollary 2 *a*.—$g_x^2/S_x^2 \geqslant 4/n$, *where*

$$g_x = 2 \sum_{i=1}^{n} \sum_{j=i+1}^{n} (x_j - x_i)/n\,(n-1);$$

with equality when $x_1 = x_2 = \ldots = x_{n-1}$, *or when* $x_2 = x_3 = \ldots = x_n$.

Proof.—Apply Corollary 2 *a* to the situation where X and Y each take each of the values $x_1, \ldots, x_n$ with probability $1/n$; $p = 1/n$. One has

$$E\,|\,Y - X\,| = \sum_{i=1}^{n} \sum_{j=1}^{n} |\,x_j - x_i\,|/n^2$$

$$= 2 \sum_{i=1}^{n} \sum_{j=i+1}^{n} (x_j - x_i)/n^2 = (n-1)\,g_x/n.$$

Then

$$g_x^2/S_x^2 = n\,[E\,|\,Y - X\,|]^2/(n-1)\,[VX] \geqslant 4/n;$$

with equality when $x_1 = x_2 = \ldots = x_{n-1}$ or when $x_2 = \ldots = x_n$.

As to (3), one observes that if X is distributed over [0, 1], VX assumes its maximum value $\frac{1}{4}$ when $Pr\{X = 0\} = Pr\{X = 1\} = \frac{1}{2}$. Thus $(x_n - x_1)^2/S_x^2 \geqslant 4\,(n-1)/n$, with equality achieved when n is even and $x_1 = x_2 = \ldots = x_n x_{/2},_n\, x_{(n/2)+1} = \ldots =$.

At the suggestion of Dr. Nair, the following remarks are added, bearing on the greatest possible values of the ratios L/S_x for (1) and (3) above. We remark first that the ratio in the lemma is not greater than $h + a^2/c$ if $0 < c \leqslant a$, $h > 0$, with equality when $Z \equiv c/a$. To see this, let λ denote the ratio, and set $z = EZ$. Since $EZ^2 \geqslant (EZ)^2$, we have $\lambda \leqslant (a + hz)^2/(c + hz^2)$; but this latter assumes its maximum, $h + a^2/c$, when $z = c/a$. Under the hypotheses of Corollary 1 one then finds $[EX]^2 \leqslant q\,EX^2$ with equality when $Pr\{X = 0\} = p$, $Pr\{X = 1\} = q$. From this the method of Corollary 1 *a* yields Dr. Nair's maxima (1 *a*) and (1 *b*) in [3] (obtained by him in [1]). Also, using the method of Corollary 1, one finds

$$VX = EX^2 - (EX)^2 = p + (q - p)\,ET^2 - [p + (q - p)\,ET]^2$$
$$\geqslant pq - 2p\,(q - p)\,(ET)\,[1 - ET] \leqslant p/2,$$

with equality when $T \equiv \frac{1}{2}$. This gives Dr. Nair's maximum (2) in [3] (obtained by him in [1]).

References

1. Nair, K. R. .. "Certain symmetrical properties of unbiased estimates of variance and covariance," *Jour. Ind. Soc. Agric. Stat.*, 1948, **1**, 162–72.
2. ——— .. "A note on the estimation of mean rate of change," *ibid.*, 1956, **8**, 122–24.
3. ——— .. "A tail-piece to Brunk's paper," *Ibid.* (Present issue).

A Tail-Piece to Brunk's Paper

By K. R. Nair

My correspondence with Dr. H. D. Brunk to which he adverts in his paper, appearing elsewhere in this issue, revealed that he had not seen my (1948) paper. It will be of interest to readers of his paper if the lower limits obtained by him for certain ratios are placed side by side with the corresponding upper limits I had given in my paper.

Thus we have,

$$\frac{1}{n} \leqslant \frac{(x_n - \bar{x})^2}{S_x^2} \leqslant \frac{(n-1)^2}{n} \tag{1 a}$$

$$\frac{1}{n} \leqslant \frac{(\bar{x} - x_1)^2}{S_x^2} \leqslant \frac{(n-1)^2}{n} \tag{1 b}$$

$$\frac{4(n-1)}{n} \leqslant \frac{(x_n - x_1)^2}{S_x^2} \leqslant 2(n-1) \tag{2}$$

$$\frac{4}{n} \leqslant \frac{g_x^2}{S_x^2} \leqslant \frac{4(n+1)}{3n}. \tag{3}$$

In my (1948) paper it was shown that the upper limit in (1 *a*) is reached when $x_1 = x_2 = \ldots = x_{n-1}$; that in (1 *b*) when $x_2 = \ldots = x_n$; and that in (2) when $x_2 = \ldots = x_{n-1} = \frac{1}{2}(x_1 + x_n)$. The (1948) and (1956) papers show by two different methods that the upper limit in (3) is reached when $x_1, x_2, \ldots, x_n$ are equally spaced.

It follows from our combined results that when $(x_n - \bar{x})/S_x$ reaches its upper limit, $(\bar{x} - x_1)/S_x$ reaches its lower limit, and *vice versa*; and that, in either case, g_x/S_x reaches its lower limit.

Finally, I wish to thank Dr. Brunk for coming forward with a solution of the problems posed by me and for sending me an advance copy of the manuscript of his paper.

REFERENCE

Brunk, H. D. .. "Note on two papers of K. R. Nair," *J. Ind. Soc. Agric. Stat.* (Present issue).